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Plenary talk: p-adic orders of Tribonacci numbers

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Abstract

The Tribonacci numbers $(T_n)_{n\geq 0}$ satisfy the recursion $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for $n \geq 0$ with $T_0 = 0$, $T_1 = T_2 = 1$. Marques and Lengyel (2014) obtained a nice formula for the 2-adic order of the "Tribonacci" numbers. They optimistically conjectured that a similar formula holds true for every prime p. In this talk, we will see that their conjecture was indeed far too optimistic; in particular, it fails for all but seven primes below 600. We will also present some ideas concerning when one may expect a nice formula to hold and when not for very general linearly recurrent sequences, not only for the Tribonacci numbers.

Coauthors: Y. Bilu, J. Nieuwveld, J. Ouaknine, D. Purser and J. Worrell.

On properties of sequences satisfying the Delannoy recurrence

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Abstract

A Fibonacci Quarterly problem of years past used an interesting combinatorial identity in a solution. The proof of the identity led to the discovery of a sequence satisfying the Delannoy recurrence ($D(m, n) = D(m-1, n) + D(m, n-1) + D(m-1, n-1)$) that was not the well-known Delannoy numbers. Generalizing this sequence, a family of related sequences satisfying the Delannoy recurrence was defined as the Generalized Delannoy Numbers. We examine these to discover properties shared amongst any sequence satisfying the Delannoy recurrence.

Coauthor: Steven Edwards

Random 2-cell embeddings of graphs in surfaces

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Abstract

A random 2-cell embedding of a connected graph G in orientable surfaces is obtained by choosing a random local rotation around each vertex. Under this setup, the number of faces or the genus of the corresponding 2-cell embedding becomes a random variable. Random embeddings of a bouquet of n loops and those of n parallel edges connecting two vertices have been extensively studied and are well-understood (see [Lando & Zvonkin, Graphs on surfaces and their applications, Springer 2004]).

In his breakthrough work ([Stahl, Permutation-partition pairs, JCTB 1991] and a series of other papers), Stahl developed the foundation of "random topological graph theory". Most of his results have been unsurpassed until today. Improving Stahl's results, we prove that the expected number of faces in a random embedding of any graph is at most linear in the number of vertices of the graph. Linear graph families, which have been studied extensively in the past, give rise to direct recurrence relations and they show that our linear bound is essentially best possible. In contrast to this, we show that the expected number of faces of almost all n-vertex graphs of bounded degree is $\Theta(\log n)$.

Coauthors: Jesse Campion Loth, Kevin Halasz, Tomas Masarik and Robert Samal

Products of two repdigits in some recurrence sequences

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Abstract

In this talk we consider the problem of searching for all Padovan and Perrin numbers which can be expressed as a product of two repdigits in the base b, where $2 \leq b \leq 10$. Padovan sequence $(P_n)_{n\geq 0}$ is defined with $P_0 = P_1 = P_2 = 1$ and $P_{n+3} = P_{n+1} + P_n$ for $n \geq 0$, while Perrin sequence $(T_n)_{n\geq 0}$ is given by $T_0 = 3$, $T_1 = 0$, $T_2 = 2$ and the same recurrence relation $T_{n+3} = T_{n+1} + T_n$ for $n \geq 0$. We prove that the largest Padovan and Perrin numbers which can be expressed as a product of two repdigits are $P_{25} = 616 = 77_{10} \cdot 8_{10}$ and $T_{22} = 486 = 22_8 \cdot 33_8 = 11_8 \cdot 66_8$. In the proofs, we use some tools from Diophantine approximation and Baker?s theory on linear forms in logarithms of algebraic numbers.

Coauthors: Kouessi Norbert Adédji, Alain Togbé

Counting on Fibonacci Polyominoes and Fibonacci Graphs

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Abstract

We introduce the k-Fibonacci polyominoes, a new family of polyominoes associated to the binary words avoiding k consecutive 1's, also called, generalized k-Fibonacci words. We present several formulas for the generating functions enumerating k-Fibonacci polyominoes according to the area, semi-perimeter, and to the number of lattice vertices. We also introduce the k-Fibonacci graphs, then we obtain the generating function for the total number of vertices and edges, the distribution of the degrees, and the total number of k-Fibonacci graphs that have a Hamiltonian cycle.

Divisibility properties of some generalized Fibonacci numbers

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Abstract

The *determinant Hosoya triangle* H, is a triangular array where the entries (points) are the determinants of two-by-two Fibonacci matrices. In this talk we discuss the primality of the entries of the triangle H. We present some divisibility properties of this type of numbers, give several examples, and give an abundance of data indicating a high density of primes in H. Since the entries of the triangle may also be expressed as generalized Fibonacci numbers, the abundance of prime numbers in the triangle prompts the question; under what conditions is a generalized Fibonacci number a prime number? In particular, under what conditions a point of H is a prime number?

Coauthors: H. Ching, A. Mukherjee, and J. C. Saunders

A conjecture on Pisot numbers

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Abstract

Let $\alpha > 1$ be a real number. Define a sequence of integers $(x_n)_{n\geq 0}$ by $x_0 \geq 1$ and $x_{n+1} = \lfloor \alpha x_n \rfloor$. We conjecture that (x_n) is a linear recurrent sequence if and only if α is a Pisot number. We will present work that supports the conjecture (unless we have a full proof by J-day).

Generalizing Arndt's pairwise descent integer compositions

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Abstract

In 2013, Joerg Arndt recorded that the Fibonacci numbers count a type of integer composition, where the first part is greater than the second, the third part is greater than the fourth, etc. We provide two combinatorial proofs of this result. Also, we consider various generalizations and establish families of related recurrence relations.

Coauthor: Aram Tangboonduangjit

Exact divisibility by powers of some binary numbers

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Abstract

Recall that for integers $d \geq 2$, $k \geq 0$, and $a \geq 1$ we say that d^k exactly divides a and write $d^k \parallel a$ if $d^k \mid a$ and $d^{k+1} \nmid a$. We obtain certain divisibility and exact divisibility results for the powers of the balancing and Lucas-balancing numbers applying the concept of p-adic valuation of these numbers. In particular, we show that $B_n^k \parallel m$ if and only if $B_n^{k+1} \parallel B_{nm}$ for all $m, n \geq 2$ and $k \geq 1$ and $C_n^k \parallel m$ if and only if $C_n^{k+1} \parallel C_{nm}$ for all $m, n \geq 2$ and $k \geq 1$. In addition, we obtain similar results for the Pell and associated Pell numbers.

Coauthors: Gopal Krishna Panda

Random Differences in the Number of Summands of Zeckendorf Decompositions

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Abstract

Zeckendorf's Theorem states that any non-negative integer m can be uniquely written as the sum of non-consecutive Fibonacci numbers $\{F_n\}$; this sum is called the *Zeckendorf decomposition* of m. It is natural to ask how many summands appear in the Zeckendorf decomposition of a given integer m, which is denoted by $Z(m)$. Lekkerkerker proved that the average value of $Z(X_n)$ where X_n is uniformly distributed in $[F_n, F_{n+1})$ is $\frac{1}{\phi+2}n + O(1)$. Kologlu, Kopp, Miller, and Wang expanded on Lekkerkerker's expectation result, showing that the normalized distribution of $Z(X_n)$ converges to a Gaussian as $n \to \infty$. In a similar spirit, Beckwith et al. investigated the statistic of gaps between the indices of summands in a Zeckendorf decomposition. They proved that the probability that the Zeckendorf decomposition of X_n has an index gap of length j exhibits a geometric decay with ratio ϕ^{-2}. These results hold not only for the Fibonacci sequence, but also for other, more general sequences known as *positive linear recurrence sequences*, or PLRS. We continue the study of Zeckendorf and PLRS-based decompositions. One of the new statistics we investigate is the behavior of the random variable $T(X,Y) = Z(X) + Z(Y) - Z(X+Y)$. If $X, Y \in [F_n, F_{n+1})$ are iid uniformly distributed, then our moment computation suggests that the statistic T is asymptotically normal. A key ingredient in understanding the distribution of T is to model it via a stochastic process. Using first step analysis and the central limit theorem for weakly dependent random variables, we are able to prove that the normalized statistic T tends to a Gaussian as $n \to \infty$.

Coauthors: Guilherme Zeus Dantas e Moura, Xuyan Liu, Wyatt Milgrim, Steven J. Miller, Prakod Ngamlamai

A note on bi-periodic incomplete Horadam numbers

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Abstract

There are several combinatorial interpretations of Fibonacci numbers, one among them is the $(n+1)$th Fibonacci number counts the number of tilings of an n-board using squares and dominoes which can be expressed as:

$$F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}.$$

By using this well known combinatorial expression, the incomplete Fibonacci numbers are defined as:

$$F_n\left(k\right) = \sum_{i=0}^{k} \binom{n-1-i}{i}, \; 0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

In this talk, we consider the bi-periodic Horadam sequence $\{w_n\}$, which is defined by the recurrence relation:

$$w_n = a^{\xi(n+1)} b^{\xi(n)} w_{n-1} + c w_{n-2}, \; n \geq 2,$$

with arbitrary initial values w_0, w_1. Here $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function of n and a, b, c are nonzero real numbers. We introduce bi-periodic incomplete Horadam numbers which give a natural generalization of incomplete Fibonacci numbers and we give recurrence relations, generating function, and some basic properties of them.

Coauthors: Amine Belkhir, Mehmet Dağlı

On the 2-adic valuation of generalized Fibonacci sequences

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Abstract

Let $F_n^{(k)} = F_n$ denote the generalized Fibonacci number of order k defined by the recurrence $F_n = F_{n-1} + F_{n-2} + \cdots + F_{n-k}$, with initial conditions $F_1 = 1$ and $F_n = 0$ for $-(k-2) \leq n \leq 0$. In the light of recent study of such sequences, we characterize the 2-adic valuation of the sequences $(F_n^{(k)})$, and draw some conclusions concerning their zero-multiplicity. In addition to the theory of 2-adic analytic functions, our method also incorporates an adaptation of a classical formula of Ferguson.

The Markoff conjecture for Fibonacci numbers

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Abstract

The Markoff conjecture states that any ordered solution triple (a, b, c) of the Markoff equation

$$a^2 + b^2 + c^2 = 3abc$$

is uniquely determined by its largest element. That is, if (a_1, b_1, c) and (a_2, b_2, c) are two Markoff solution triples with $a_i \leq b_i \leq c$, $i = 1, 2$, then $a_1 = a_2$ and $b_1 = b_2$. This conjecture has been around for more than a hundred years and despite the growing number of mathematicians that have taken an interest in this problem, there has been little significant progress. A variety of non trivial methods have been used to prove the conjecture in the case of prime powers. However, the second author gave a completely elementary proof of this case which is based on the identity that if (a_1, b_1, c) and (a_2, b_2, c) are two Markoff triples, then

$$(a_1a_2 - b_1b_2)(a_1b_2 - b_1a_2) = c^2(a_1b_1 - a_2b_2).$$

Can this identity be used to tackle the general case? We are interested in the Markoff Fibonacci triples; it is known that if F_n is any odd indexed Fibonacci number, then $(1, F_{n-2}, F_n)$ is a Markoff triple. In this presentation we propose new lines of approach to look into the conjecture for F_n, using the properties of Fibonacci numbers and the identity given above.

Coauthor: Steven J. Miller

Generalized Fibonacci numbers and random processes

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Abstract

We introduce a new generalization of Fibonacci numbers based on the probabilistic approach and concepts of combinatorics on words associated with it, and find a link with the appropriate random process. We find the generating function for the numbers of the generalized Fibonacci sequence and then find its relationship with the generating function of the random process. By studying certain queuing system with Poisson distribution, we obtain under certain conditions the probability distribution that involves Fibonacci numbers. Using this distribution we present a probabilistic derivation of the generating function of the Fibonacci numbers. We also obtain double series identities for the Fibonacci numbers that involve binomial coefficients, from which the Catalan and Lucas formulas follow.

The Meta-C-Finite Ansatz

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Abstract

The Fibonacci numbers $F(n)$ satisfy the famous recurrence $F(n+2) = F(n+1) + F(n)$. The "C-finite ansatz" tells us that the family of sequences $F(2n), F(3n), F(4n), \dots,$ along with their sums and products satisfy similar recurrences. However, even more is true. We will show that the recurrences satisfied by $F(ni)$ and $F(ni)F(nj)$, for any C-finite sequence F, satisfy meta recurrences which lead to generating function and summation identities.

Age structured Fibonacci Circle Patterns

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Abstract

The Fibonacci sequence patterns found in many types of multicellular organisms involve a radial structure. While the phyllotactic processes and spatiotemporal patterns involved in formation of branched and geometric capitulum structures have been extensively studied, the processes and patterns involved in forming circle structures have been understudied. The beginning of tissue development often starts with an initiating cell that, through one or more of its cell divisions, generates offspring cell populations that form ring or circular structures. Accordingly, I created a model for age structured Fibonacci circle pattern formation. The model was designed using the rules for circular pattern formation and circumferential growth based on Parkinson's Law. The age structure is modeled according to a cell age-label, number of initiating cells, the direction of cell division, the order of cell divisions, and the specific number of cell divisions that the offspring cells undergo. The patterns generated from the sequences of cell divisions were then analyzed for a fit to various Fibonacci sequences and known angles of divergence in phyllotaxis. Modeling results show that different age-structured Fibonacci circle patterns fit different Fibonacci sequences (OEIS numbers A000045, A000032, A000285, A022095, A022096, A022097, A013655, A022113, A022114) and p-Fibonacci numbers (A000930, A003269, A003520, A005708). That specific age-structured Fibonacci circle patterns fit different Fibonacci sequences indicates that a simple set of rules controls development of tissue structure and organization in plants and animals. Conclusion: Studying Fibonacci circle patterns indicates that the mechanism underlying the organization of biological tissues involves a specific age-structure rule for cells in each tissue.

On k-regularity of valuations and last nonzero digits of linear recurrence sequences

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Abstract

Let $(s_n)_{n\geq 0}$ be a linear recurrence sequence of integers. For a fixed integer base $b \geq 2$ we consider the sequences $(\nu_b(s_n))_{n\geq 0}$ and $(\ell_b(s_n))_{n\geq 0}$ of "b-adic valuations" and last nonzero digits in base b expansion of s_n, respectively. We assume that for each prime factor p of b the sequence $(s_n)_{n\geq 0}$ can be interpolated along arithmetic progressions by a collection of p-adic analytic functions. The main goal of the talk is to give a classification concerning k-regularity of the sequences $(\nu_b(s_n))_{n\geq 0}$ and $(\ell_b(s_n))_{n\geq 0}$ in terms of the roots of said p-adic functions in $\mathbb{Z}_p$. In fact, we provide a more general result, which extends a theorem obtained by Shu and Yao for b prime. As an application, we strengthen a result by Murru and Sanna on b-adic valuations of Lucas sequences of the first kind. Moreover, we show how to determine precisely which terms of these sequences can be represented as a sum of three squares.

Some Properties of Fibonacci-Pascal Triangle

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Abstract

This present work will introduce a triangular array defined from the combination of the Hosoya triangle (Fibonacci triangle) and Pascal's triangle. The sum of all elements in each row, the expression of each entry, and the generalization of the triangular array with k-generalized Fibonacci numbers are investigated and determined.

Coauthors: Colin Leyner and Kate Tanner

The Fibonacci Sequence and Math Outreach

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Abstract

For the past decade the speaker and his colleagues have created games and activities involving the Fibonacci numbers and brought these to local elementary schools, running them in multiple grades from K through 6. We have found kids as young as 5 are able to discover the Fibonacci relation and some properties from a rectangle tiling game inspired by the Fibonacci spiral. We report on some of the methods used in the classrooms, and the active learning outcomes.
Coauthors: Anna Mello, Cameron and Kayla Miller

On k–generalized Fibonacci Diophantine triples

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Abstract

For $k \geq 2$, the k-generalized Fibonacci sequence $\left(F_n^{(k)}\right)$ is defined by the initial values $0, 0, \ldots, 0, 1$ (k terms) and each term afterwards is the sum of the k preceding terms. In this paper, we prove that the system

$$\begin{aligned} ab + 1 &= F_x^{(k)} \\ ac + 1 &= F_y^{(k)} \\ bc + 1 &= F_z^{(k)} \end{aligned}$$

has no solution for $1 < a < b < c$ with $a \leq 10^3$.

On Zeckendorf Related Partitions Using the Lucas Sequence

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Abstract

Zeckendorf proved that every positive integer has a unique partition as a sum of nonconsecutive Fibonacci numbers. Similarly, every natural number can be partitioned into a sum of nonconsecutive terms of the Lucas sequence, although such partitions need not be unique. In this talk, we show that a natural number can have at most two distinct nonconsecutive partitions in the Lucas sequence and calculate the limiting value of the proportion of natural numbers that are not uniquely partitioned into the sum of nonconsecutive terms in the Lucas sequence.

Coauthors: Hung V. Chu and Steven J. Miller

On 1's in continued fraction expansions of square roots of prime numbers

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Abstract

The aim of the talk is to present results concerning appearance of consecutive 1's in expansions of square roots of prime numbers into continued fractions. Under assumption of Hypothesis H of Schinzel and Sierpiński I will give results on frequency of appearance of 1's in expansions of square roots of prime numbers into continued fractions. In order to do this I will apply some generalization of Cassini's identity. Next, using the theorem that the sequence of square roots of consecutive prime numbers is equidistributed modulo 1 and the facts from the measure theory of continued fractions, I will present results on the asymptotic densities (with respect to the set of prime numbers) of the sets of prime numbers with prescribed (not necessarily beginning) terms in expansions of their square roots into continued fractions. In particular I will give the exact values of asymptotic densities for sets $\mathcal{A}_k$ of prime numbers such that the periods of expansions of their square roots into continued fractions begin with k 1's but not $k+1$ 1's.

The talk is based on a joint work with Maciej Ulas.

GCD of sums of k consecutive Fibonacci, Lucas, and generalized Fibonacci numbers

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Abstract

We explore the sums of k consecutive terms in the generalized Fibonacci sequence $(G_n)_{n\geq 0}$ given by the recurrence $G_n = G_{n-1} + G_{n-2}$ for all $n \geq 2$ with integral initial conditions G_0 and G_1. In particular, we give precise values for the greatest common divisor (GCD) of all sums of k consecutive terms of $(G_n)_{n\geq 0}$. When $G_0 = 0$ and $G_1 = 1$ (respectively, $G_0 = 2$ and $G_1 = 1$), we yield the GCD of all sums of k consecutive Fibonacci (respectively, Lucas) numbers. Denoting the GCD of all sums of k consecutive generalized Fibonacci numbers by the symbol $\mathcal{G}_{G_0,G_1}(k)$, we give two tantalizing characterizations for these values, one involving a simple formula in k and another involving generalized Pisano periods:

$$\mathcal{G}_{G_0,G_1}(k) = \gcd(G_{k+1} - G_1,\, G_{k+2} - G_2)$$

and

$$\mathcal{G}_{G_0,G_1}(k) = \operatorname{lcm}\{m \mid \pi_{G_0,G_1}(m) \text{ divides } k\},$$

where $\pi_{G_0,G_1}(m)$ denotes the generalized Pisano period of the generalized Fibonacci sequence modulo m. The fact that these vastly different-looking formulas coincide leads to some surprising and delightful new understandings of the Fibonacci and Lucas numbers. Generalizing this research project to the GCD of the sum of k consecutive squares of generalized Fibonacci numbers has very recently been completed in joint work with myself and Dr. Jürgen Spilker (professor emeritus of the University of Freiburg). However, in this talk we will focus only on the topic above.

Coauthor: Dan Guyer

Sums of Reciprocals of Recurrence Relations

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Abstract

There is a growing literature on sums of reciprocals of polynomial functions of recurrence relations with constant coefficients and fixed depth, such as Fibonacci and Tribonacci numbers, products of such numbers, and balancing numbers (numbers n such that the sum of the integers less than n equals the sum of the r integers immediately after, for some r which is called the balancer of n; If n is included in the summation, we have the cobalancing numbers, and r is called the cobalancer of n). We generalize previous work to reciprocal sums of depth two recurrence sequences with arbitrary coefficients and the Tribonacci numbers, and show our method provides an alternative proof of some existing results.

We define (a, b) balancing and cobalancing numbers, where a and b are constants that multiply the left-hand side and right-hand side respectively, and derive recurrence relations describing these sequences. We show that for balancing numbers, the coefficients $(3, 1)$ is unique such that every integer is a $(3, 1)$ balancing number, and proved there does not exist an analogous set of coefficients for cobalancing numbers. We also found patterns for certain coefficients that have no balancing or cobalancing numbers.

This work is joint with Bella Cui, Hao Cui, Jack Hu, Lisa Liu, Joyce Qu and Fengping Ren

Coauthors: Sophia Davis, Irfan Durmic, Steven Miller and Alicia Smith Reina

On the Hurwitz-type zeta function associated to the Lucas sequence

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Abstract

We study the theta function and the Hurwitz-type zeta function associated to the Lucas sequence $U = \{U_n(P,Q)\}_{n\geq 0}$ of the first kind determined by the real numbers P, Q under certain natural assumptions on P and Q. We deduce an asymptotic expansion of the theta function $\theta_U(t)$ as $t \downarrow 0$ and use it to obtain a meromorphic continuation of the Hurwitz-type zeta function $\zeta_U(s,z) = \sum_{n=0}^{\infty} (z+U_n)^{-s}$ to the whole complex $s-$plane. Moreover, we identify the residues of $\zeta_U(s,z)$ at all poles in the half-plane $\Re(s) \leq 0$.
Coauthors: Lejla Smajlović and Zenan Šabanac

On the Generalized Tribonacci Zeta Functions and its Meromorhic Continuation

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Abstract

We obtain a meromorphic continuation of the generalized Tribonacci zeta function to the whole plane. The values of the some generalized Tribonacci zeta functions at negative integers are computed.

Coauthors: Lejla Smajlović and Lamija Šćeta

The Self-Counting Flow

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Abstract

Our talk is based on the paper "The self-counting identity", published in the *Fibonacci Quarterly* in May 2017, vol. 55 and can be considered as its continuation.
In the beginning, we define the "self-counting flow Φ", which represents a tool for getting from one positive integer sequence to a corresponding other one. It is - so to say - a flow on all positive integer sequences and thereby the self-counting sequence $\{a_k\}_{k=1}^{\infty} = \{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots\}$ shows itself as a unique fixed point.
Various methods allow us to study the properties of the flow Φ, such as its trajectories and the attraction of its fixed point. We also examine if the self-counting sequence $\{a_k\}_{k=1}^{\infty}$ is the point of convergence of each positive integer sequence under the flow Φ.
At the end, we show some properties of other flows on positive integer sequences, for example those of the "Fibonacci flow $\mathcal{F}$".

Some consequences of a theorem of Reznick

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Abstract

Let (a_n) denote Stern's sequence defined recursively by $a_1 = 1, a_{2n} = n, a_{2n+1} = a_n a_{n+1}$. In 2006, Resnick published a formula for the asymptotic densities of the sets $\{n : a_n \equiv a \pmod m\}$. We sketch how Markov chains help derive this formula. Using this formula we show that, for fixed $m \geq 1$, the sequence $\left\lfloor \frac{a_n}{m} \right\rfloor$ is uniformly distributed modulo m^k for every k. Next, we indicate how to extend these results to the sequences $(a_n a_{n+1})$ and, more generally, $Q(a_n, a_{n+1})$ where Q is a primitive integer quadratic form (and m is relatively prime to the discriminant of Q). Finally, we consider R_n, the number of ways of representing n as a sum of distinct Fibonacci numbers and indicate a proof that, for any m, R_n is a multiple of m for "almost all" n.

A System of Four simultaneous Recursions Generalization of the Ledin-Shannon-Ollerton Identity

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Abstract

This paper further generalizes a recent result of Shannon and Ollerton who resurrected an old identity due to Ledin. This paper generalizes the Ledin-Shannon-Ollerton result to all the metallic sequences. The results give closed formulas for the sum of products of powers of the first n integers with the first n members of the metallic sequence.

Three key innovations of this paper are i) reducing the proof of the generalization to the solution of a system of 4 simultaneous recursions; ii) use of the shift operation to prove equality of polynomials; and iii) new OEIS sequences arising from the coefficients of the four polynomial families satisfying the 4 simultaneous recursions.

On perfect powers that are sum of two balancing numbers

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Abstract

Let B_k denote the k^{th} term of balancing sequence. In this paper we find all positive integer solutions of the Diophantine equation $B_n + B_m = x^q$ in variables (m, n, x, q) under the assumption $n \equiv m \pmod 2$. Furthermore, we study the Diophantine equation

$$B_n^3 \pm B_m^3 = x^q$$

with positive integer $q \geq 3$ and $\gcd(B_n, B_m) = 1$.

Coauthors: Sudhansu Sekhar Rout, Gopal Krishna Panda

On p-adic continued fractions and periodic representations of quadratic irrationals

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Abstract

First of all, we provide a brief overview about the definition of continued fractions in the field of p-adic numbers $\mathbb{Q}_p$. Currently there is no standard algorithm, since it is hard to replicate all the good properties that continued fractions have over the real numbers regarding best approximations, finiteness and periodicity. In particular, there does not exist a definition of p-adic continued fractions such that the analogue of Lagrange's theorem holds. In this talk, we will focus on the Browkin's approach and we will see some results about the periodicity of Browkin's continued fractions. Then, we will present a periodic representation for any quadratic irrational via p-adic continued fractions, even if it is not obtained by a specific algorithm. This periodic representation provides simultaneous rational approximations for a quadratic irrational both in $\mathbb{R}$ and $\mathbb{Q}_p$. We will conclude with an example involving the Golden mean.

An Infinite 2-Dimensional Array Associated With Electric Circuits

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Abstract

Except for Koshy who devotes seven pages to applications of Fibonacci Numbers to electric circuits, most books and the Fibonacci Quarterly have been relatively silent on applications of graphs and electric circuits to Fibonacci numbers. This paper continues a recent trend of papers studying the interplay of graphs, circuits, and Fibonacci numbers by presenting and studying the Circuit Array, an infinite 2-dimensional array whose entries are electric resistances labelling edge values of circuits associated with a family of graphs. The Circuit Array has several features distinguishing it from other more familiar arrays such as the Binomial Array and Wythoff Array. For example, it can be proven modulo a strongly supported conjecture that the numerators of its left-most diagonal do not satisfy any linear, homogeneous, recursion, with constant coefficients (LHRCC). However, we conjecture with supporting numerical evidence an asymptotic formula involving π satisfied by the left-most diagonal of the Circuit Array.

Coauthor: Emily J. Evans

On the Diophantine equation $N_n = x^a \pm x^b + 1$

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Abstract

Narayana's cows sequence $\{N_n\}_{n\geq 0}$ is a ternary recurrent sequence given by the recurrence relation $N_{n+3} = N_{n+2} + N_n$ with seeds $N_0 = 0, N_1 = 1, N_2 = 1$. In this talk, we are interested to find Narayana numbers of the form $x^a \pm x^b + 1$. In particular, we solve the exponential Diophantine equation $N_n = x^a \pm x^b + 1$, where a, b are nonnegative integers with $0 \leq b < a$ and $2 \leq x \leq 30$. For the proof of the said result, we use lower bounds for linear forms in logarithms (Baker's theory) and a version of Baker-Davenport reduction method in Diophantine approximation.

Coauthor: Prasanta Kumar Ray

A Problem on the Cardinality of Difference Sequences

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Abstract

At the 19th International Conference on Fibonacci Numbers and their Applications, Clark Kimberling proposed the following open problem: Let r be an irrational number with fractional part between $\frac{1}{3}$ and $\frac{1}{2}$. Let C_n be the number of distinct n^{th} differences of the sequence $(\lfloor kr \rfloor)$, where $k \in \mathbb{N}$. Prove or disprove that $C_n = (2, 3, 3, 5, 4, 7, 5, 9, 6, 11, 7, 13, 8, 15, 9, \ldots)$, which is simply a riffle of $(2, 3, 4, 5, 6, \ldots)$ and $(3, 5, 7, 9, 11, \ldots)$. In this talk I will discuss the problem and provide some interesting results.

Coauthors: Danielle Cox, Shayne Breen

Closed forms of rationally weighted binomial sums via calculus methods

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Abstract

In this note, we apply calculus methods to find closed forms of some rationally weighted binomial sums and several generalizations. As consequences of our approach, many identities, like [W. J. Greenstreet, *Problem AL*-195, American Math. Monthly, Vol. 11, No. 3 (Mar., 1904), p. 73], [S. Epsteen, *Problem AL*-218, American Math. Monthly, Vol. 11, No. 12 (Dec., 1904), p. 240], [B. F. Finkel, *Problem AL*-320, American Math. Monthly, Vol. 16, No. 3 (Mar., 1909), p. 55.] or [T. Koshy, Fibonacci and Lucas Numbers with Applications - Chapter 25 (2nd Ed.), 2018], can be easily shown by our method. In the process, we recover the (generalized) Stirling numbers of the second kind.

Coauthors: Andreea M. Stanica, Gabriela N. Stanica

Q-bonacci words and numbers

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Abstract

We present a quite curious generalization of multi-step Fibonacci numbers. For any positive rational q, we enumerate binary words of length n whose maximal factors of the form $0^a 1^b$ satisfy $a = 0$ or $aq > b$. When q is an integer we rediscover classical multi-step Fibonacci numbers: Fibonacci, Tribonacci, Tetranacci, etc. When q is not an integer, obtained recurrence relations are connected to certain restricted integer compositions. We also discuss Gray codes for these words, and a possibly novel generalization of the golden ratio.

Counting on the Hosoya Triangle

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Abstract

We provide a tiling interpretation of the entries of Hosoya's Triangle, leading to combinatorial proofs of many identities.
Coauthor: Daniela Elizondo

Fence tiling derived identities involving the metallonacci numbers squared or cubed

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Abstract

We refer to the generalized Fibonacci sequences defined by $M_{n+1}^{(c)} = cM_n^{(c)} + M_{n-1}^{(c)}$ for $n > 1$ with $M_1^{(c)} = 1$ for $c = 1, 2, \ldots$ as the c-metallonacci numbers. We consider the tiling of an n-board (a $n \times 1$ rectangular board) with c colours of $1/p \times 1$ tiles (with the shorter sides always aligned horizontally) and $(1/p, 1-1/p)$-fence tiles for $p \in \mathbb{Z}^+$. A (w, g)-fence tile is composed of two $w \times 1$ sub-tiles separated by a $g \times 1$ gap. The number of such tilings equals $(M_{n+1}^{(c)})^p$ and we use this result for the cases $p = 2, 3$ to devise straightforward combinatorial proofs of identities relating the metallonacci numbers squared or cubed to other combinations of metallonacci numbers. Special cases include relations between the Pell numbers cubed and the even Fibonacci numbers. Most of the identities derived here appear to be new.
Coauthor: Kenneth Edwards

Further results on restricted combinations via restricted-overlap tiling with combs

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Abstract

We consider $S_{n,k}$, the number of k-subsets of $\{1, 2, \ldots, n\}$ such that no two elements of the subset differ by any element of a given set $\mathcal{Q}$ whose largest element is q. We obtain recursion relations for $S_{n,k}$ for various instances of $\mathcal{Q}$ by using a bijection between such subsets and the restricted-overlap tilings of an $(n+q) \times 1$-board with unit squares and k combs. The combs employed here are composed of a number of integer-length sub-tiles (called teeth) separated by integer-length gaps. The lengths of the teeth and gaps depend on $\mathcal{Q}$. All results obtained appear to be new.

Zeckendorf representation of multiplicative inverses modulo a Fibonacci number

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Abstract

Let $(F_n)_{n\geq 1}$ be the sequence of Fibonacci numbers, which is defined by the initial conditions $F_1 = F_2 = 1$ and by the linear recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Every positive integer n can be written as a sum of distinct non-consecutive Fibonacci numbers, that is, $n = \sum_{i=1}^{m} d_i F_i$, where $m \in \mathbb{N}$, $d_i \in \{0, 1\}$, and $d_i d_{i+1} = 0$ for all $i \in \{1, \ldots, m-1\}$. This is called the *Zeckendorf representation* of n and, apart from the equivalent use of F_1 instead of F_2 or vice versa, is unique.
The Zeckendorf representation of integer sequences has been studied in several works. For instance, Filipponi and Freitag studied the Zeckendorf representation of numbers of the form F_{kn}/F_n, F_n^2/d and L_n^2/d, where L_n are the Lucas numbers and d is a Lucas or Fibonacci number. Filipponi, Hart, and Sanchis analyzed the Zeckendorf representation of numbers of the form mF_n. Filipponi determined the Zeckendorf representation of mF_nF_{n+k} and mL_nL_{n+k} for $m \in \{1, 2, 3, 4\}$. Prempreesuk, Noppakaew, and Pongsriiam determined the Zeckendorf representation of the multiplicative inverse of 2 modulo F_n, for every positive integer n not divisible by 3, where F_n denotes the nth Fibonacci number. We determine the Zeckendorf representation of the multiplicative inverse of a modulo F_n, for every fixed integer $a \geq 3$ and for all positive integers n with $\gcd(a, F_n) = 1$. Our proof makes use of the so-called base-φ expansion of real numbers.
Coauthors: Nadir Murru, Carlo Sanna

The Bergman Game

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Abstract

P. Baird-Smith A. Epstein, K. Flint, and S. J. Miller (2018) created the Zeckendorf Game, a two-player game which takes as an input a positive integer n and, using moves related to the Fibonacci recurrence relation, outputs the unique decomposition of n into a sum of non-consecutive Fibonacci numbers. Following this work and that of G. Bergman (1957), which proved the existence and uniqueness of such φ-decompositions, we formulate the Bergman Game which outputs the unique decomposition of n into a sum of non-consecutive powers of φ, the golden mean.

We then formulate Generalized Bergman Games, which use moves based on an arbitrary non-increasing positive linear recurrence relation and output the unique decomposition of n into a sum of non-adjacent powers of β, where β is the dominating root of the characteristic polynomial of the chosen recurrence relation. We prove that the longest possible Generalized Bergman game on an initial state S with n summands terminates in $\Theta(n^2)$ time, and we also prove that the shortest possible Generalized Bergman game on an initial state terminates between $\Omega(n)$ and $O(n^2)$ time. We also show a linear bound on the maximum length of the tuple used throughout the game.
Coauthors: Benjamin Baily, Justine Dell, Irfan Durmic, Henry L. Fleischmann, Isaac Mijares, Steven J. Miller, Ethan Pesikoff, Luke Reifenberg, Alicia Smith Reina, and Yingzi Yang

The Narayana Sequence in finite groups

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Abstract

In this paper, the Narayana sequence modulo m is studied. The paper outlines the definition of Narayana numbers and some of their combinatorial links with Eulerian, Catalan and Delannoy numbers and other special functions. From the definition, the Narayana orbit of a 2-generator group for a generating pair $(x, y) \in G$ is defined, so that the lengths of the period of the Narayana orbit can be examined. These yield in turn the Narayana lengths of the polyhedral group and the binary polyhedral group for the generating pair (x, y) and associated properties.
Coauthors: Engin Özkan and Anthony G. Shannon

Generalizing Zeckendorf's theorem to homogeneous linear recurrences

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Abstract

Zeckendorf's theorem states that every positive integer can be written uniquely as the sum of non-consecutive shifted Fibonacci numbers $\{F_n\}$, where we take $F_1 = 1$ and $F_2 = 2$. This has been generalized for any Positive Linear Recurrence Sequence (PLRS), which informally is a sequence satisfying a homogeneous linear recurrence with a positive leading coefficient and non-negative integer coefficients. We provide two approaches to study linear recurrences with leading coefficient zero, followed by non-negative integer coefficients, with differences between indices relatively prime (abbreviated ZLRR), via two different approaches; this talk mainly investigates the second approach. The first approach involves generalizing the definition of a legal decomposition for a PLRS found in Koloğlu, Kopp, Miller and Wang. The second approach converts a ZLRR to a PLRR that has the same growth rate. We develop the Tree Algorithm, a powerful helper tool for analyzing the behavior of linear recurrence sequences. We use it to prove a very general result that guarantees conversion between certain recurrences, and develop a method to quickly determine whether a sequence diverges to $+\infty$ or $-\infty$, given any real initial values. We also analyze the runtime of the Tree Algorithm.

Coauthors: Thomas C. Martinez, Jack Murphy, Chenyang Sun

Generalized $(c-k)$*–nacci Zeckendorf Game*

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Abstract

Zeckendorf proved that every positive integer n can be written uniquely as the sum of non-adjacent Fibonacci numbers; a similar result holds for other types of positive linear recurrence sequences. These legal decompositions can be used to construct a game that starts with a fixed integer n, and players take turns using moves relating to a given recurrence relation. The game eventually terminates in a unique legal decomposition, and the player who makes the final move wins.

For the Fibonacci game, player 2 has the winning strategy for all $n > 2$. We give a non-constructive proof that for the two-player (c, k)-nacci game, for all k and sufficiently large n, player 1 has a winning strategy when c is even and player 2 has a winning strategy when c is odd. Interestingly, the player with the winning strategy can make a mistake as early as the $c+1$ turn, in which case the other player gains the winning strategy. Furthermore, we proved that for the (c, k)-nacci game with players $p \geq c + 2$, no player has a winning strategy for any $n \geq 3c^2+6c+3$. We find a stricter lower boundary, $n \geq 7$, in the case of the three-player $(1, 2)$-nacci game. We then extended the result from the multiplayer game to multialliance games, showing which alliance has a winning strategy or when no winning strategy exists for some special cases of multialliance games.

Coauthors: Steven J. Miller, Eliel Sosis, Jingkai Ye

Recursive Dynamic Model of Spiral Phyllotaxis Morphogenesis

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Abstract

The recursive dynamic model of spiral morphogenesis of phyllotaxis is based on two constant processes: each new primordium appear in the center of the inflorescence at regular intervals and all primordia grow at a constant rate. The model explains the appearance of visible spirals in the phyllotaxis pattern, the number of which is equal to the Fibonacci numbers. The operation of the model is explained with the help of the video built by the author.

Walking to Infinity on the Fibonacci Sequence

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Abstract

An interesting open problem in number theory asks whether it is possible to walk to infinity on primes, where each term in the sequence has one more digit than the previous. In this paper, we study its variation where we walk on the Fibonacci sequence. We prove that all walks starting with a Fibonacci number and the following terms are Fibonacci numbers obtained by appending exactly one digit at a time to the right have a length of at most two. In the more general case where we append at most a bounded number of digits each time, we give a formula for the length of the longest walk.

Coauthors: Steven J. Miller, Fei Peng, Tudor Popescu

S-Legal Index Difference Decomposition

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Abstract

In 1972, Zeckendorf showed that every positive integer may be expressed uniquely as the sum of non-consecutive Fibonacci numbers. A variation of the Fibonacci numbers is the Fibonacci Quilt sequence, which arises from tiling the plane through the Fibonacci spiral. Beginning with 1 in the center, we place integers in the squares of the spiral such that each square contains the smallest integer that may not be expressed as the sum of non-adjacent previous terms. In this case, adjacent refers to the sharing of a wall on the plane. Except for some small values, this adjacency is also captured in the differences of the indices of each square - two squares are adjacent if their indices differ by 1, 3 or 4. Similarly, the Padovan spiral, constructed from triangles, creates the analogous Padovan quilt sequence. In this case, except for small values, the triangles are adjacent to each other if their indices differ by 1 or 5.

Motivated by understanding the behavior of these quilt constructions, we consider a generalization: given a collection $\{a_i\}_{i=1}^n$ of positive integers and a finite set S of positive integers, we say that a sum $N = a_{i_1} + \cdots + a_{i_\ell}$ such that the i_j are distinct and $|i_j - i_k| \notin S$ for all $j \neq k$ is an S-Legal Index Decomposition (S-LID) of N using $\{a_i\}_{i=1}^n$. Define the S-LID sequence $\{a_i\}_{i=1}^\infty$ by letting $a_1 = 1$ and a_{n+1} be the smallest positive integer that does not have an S-LID using $\{a_i\}_{i=1}^n$. For example, the $\{1\}$-LID sequence is the Fibonacci sequence by Zeckendorf?s theorem. We demonstrate infinite families of sets S for which the S-LID sequence follows a simple recurrence relation, and put forth for future study the classification of all S-LID sequences by the properties of S.

Coauthors: Guilherme Zeus Dantas e Moura, Andrew Keisling, Astrid Lilly, Steven J. Miller, Matthew Phang, Santiago Miguel Velazquez Iannuzzelli

Generalizing Properties of Far-Difference Fibonacci Decompositions

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Abstract

Zeckendorf proved that every positive integer can be decomposed uniquely as a sum of distinct, non-adjacent Fibonacci numbers, with the condition $F_1 = 1$, $F_2 = 2$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. This decomposition can be generalized to any positive linear recursion sequence (PLRS), which is a sequence with terms defined by the recurrence relation $H_n = c_1 H_{n-1} + \cdots + c_k H_{n-k}$ with $c_i \in \mathbf{Z}_{\geq 0}$. A paper by Cordwell et al. showed that these generalized Zeckendorf decompositions are summand minimal.

If we now consider Zeckendorf-style decompositions with coefficients either ± 1, we arrive at what Hannah Alpert terms a "far-difference representation". Demontigny et al. showed that far-difference representations share many similarities with the Zeckendorf decomposition, including the uniqueness of decomposition when generalizing to PLRS?s and the asymptotic Gaussianity for the number of summands within a Fibonacci interval. It is natural to expect other intrinsic properties of Zeckendorf decomposition to be successfully lifted to the far-difference case. In particular, Alpert proved the minimality of far-difference representations, and we generalize her results to extend to PLRS?s using techniques in the Cordwell paper. Additionally, we present a "Far-Difference Game" analogous to the Zeckendorf Game introduced and analyzed in 2018 by Baird-Smith et al.

Coauthors: Justin Cheigh, Guilherme Zeus Dantas e Moura, Ryan S. Jeong, Andrew P. Keisling, Jacob Lehmann Duke, Xuyan Liu, Wyatt Milgrim, Steven J. Miller, Matthew Phang, Santiago Velazquez Iannuzzelli

On the Diophantine equation $F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4} + F_{n_5} = 2^a$

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Abstract

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. In this paper, On the one hand, we have determined all the powers of 2 which are sums of five Fibonacci numbers with a few exceptions that we characterize. On the other hand, we have established a conjecture showing that such equations admit infinitely many solutions.

Multivariate Fibonacci-Like Polynomials and Their Applications

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Abstract

The Fibonacci polynomials are defined recursively as $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$, where $f_0(x) = 0$ and $f_1(x) = 0$. We generalize these polynomials to an arbitrary number of variables with the r-Bonacci polynomial:

$$F_n^{[r]}(x_1, x_2, ...x_r) = \begin{cases} 0 & 0 \leq n < r-1 \\ 1 & n = r-1 \\ \sum_{i=1}^{r} x_i F_{n-i}^{[r]} & n \geq r \end{cases}$$

We extend several well-known results such as an explicit Binet-like formula and a Cassini-like identity. Additionally, we prove that the terms and coefficients of the r-Bonacci polynomials generate integer partitions and use this to derive a connection to ordinary Bell polynomials and an explicit sum formula given by

$$F_n^{[r]} = \sum_{\substack{\alpha_1, \alpha_2, \ldots \alpha_r \geq 0 \\ \alpha_1 + 2\alpha_2 \cdots + r\alpha_r = n-r+1}} \binom{\alpha_1 + \cdots + \alpha_r}{\alpha_1, \ldots, \alpha_r} \cdot x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r}$$

Moreover, we derive identities that relate the r-Bonacci polynomials, exponential Bell polynomials, Fubini numbers, and the Stirling numbers of the second kind. Finally, we show that $F_n^{[r]}$ is irreducible over $\mathbb{C}$ for $n \geq r \geq 3$.

Coauthor: Peikai Qi

What is the qth Fibonacci number, where q is rational?

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Abstract

The answer to the question raised in the title is the codenominator function, which is an integral-valued map. It is defined via a pair of functional equations. Many Fibonacci identities in the literature are valid for the codenominator. It is related to the recently introduced involution Jimm and to the outer automorphism of PGL(2,Z)

Coauthor: Buket Eren Gökmen

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